

# Lesson 1: Exponential Notation

## Classwork

$5^6$  means  $5 \times 5 \times 5 \times 5 \times 5 \times 5$  and  $\left(\frac{9}{7}\right)^4$  means  $\frac{9}{7} \times \frac{9}{7} \times \frac{9}{7} \times \frac{9}{7}$ .

You have seen this kind of notation before: it is called **exponential notation**. In general, for any number  $x$  and any positive integer  $n$ ,

$$x^n = \underbrace{(x \cdot x \cdots x)}_{n \text{ times}}$$

The number  $x^n$  is called  $x$  raised to the  $n^{\text{th}}$  power, where  $n$  is the **exponent** of  $x$  in  $x^n$  and  $x$  is the **base** of  $x^n$ .

### Exercise 1

$$\underbrace{4 \times \cdots \times 4}_{7 \text{ times}} =$$

### Exercise 6

$$\underbrace{\frac{7}{2} \times \cdots \times \frac{7}{2}}_{21 \text{ times}} =$$

### Exercise 2

$$\underbrace{3.6 \times \cdots \times 3.6}_{\text{times}} = 3.6^{47}$$

### Exercise 7

$$\underbrace{(-13) \times \cdots \times (-13)}_{6 \text{ times}} =$$

### Exercise 3

$$\underbrace{(-11.63) \times \cdots \times (-11.63)}_{34 \text{ times}} =$$

### Exercise 8

$$\underbrace{\left(-\frac{1}{14}\right) \times \cdots \times \left(-\frac{1}{14}\right)}_{10 \text{ times}} =$$

### Exercise 4

$$\underbrace{12 \times \cdots \times 12}_{\text{times}} = 12^{15}$$

### Exercise 9

$$\underbrace{x \cdot x \cdots x}_{185 \text{ times}} =$$

### Exercise 5

$$\underbrace{(-5) \times \cdots \times (-5)}_{10 \text{ times}} =$$

### Exercise 10

$$\underbrace{x \cdot x \cdots x}_{\text{times}} = x^n$$

**Exercise 11**

Will these products be positive or negative? How do you know?

$$\underbrace{(-1) \times (-1) \times \cdots \times (-1)}_{12 \text{ times}} = (-1)^{12}$$

$$\underbrace{(-1) \times (-1) \times \cdots \times (-1)}_{13 \text{ times}} = (-1)^{13}$$

**Exercise 12**

Is it necessary to do all of the calculations to determine the sign of the product? Why or why not?

$$\underbrace{(-5) \times (-5) \times \cdots \times (-5)}_{95 \text{ times}} = (-5)^{95}$$

$$\underbrace{(-1.8) \times (-1.8) \times \cdots \times (-1.8)}_{122 \text{ times}} = (-1.8)^{122}$$

**Exercise 13**

Fill in the blanks about whether the number is positive or negative.

If  $n$  is a positive even number, then  $(-55)^n$  is \_\_\_\_\_.

If  $n$  is a positive odd number, then  $(-72.4)^n$  is \_\_\_\_\_.

**Exercise 14**

Josie says that  $\underbrace{(-15) \times \dots \times (-15)}_{6 \text{ times}} = -15^6$ . Is she correct? How do you know?

**Problem Set**

1. Use what you know about exponential notation to complete the expressions below.

$$\underbrace{(-5) \times \dots \times (-5)}_{17 \text{ times}} =$$

$$\underbrace{3.7 \times \dots \times 3.7}_{\text{--- times}} = 3.7^{19}$$

$$\underbrace{7 \times \dots \times 7}_{\text{--- times}} = 7^{45}$$

$$\underbrace{6 \times \dots \times 6}_{4 \text{ times}} =$$

$$\underbrace{4.3 \times \dots \times 4.3}_{13 \text{ times}} =$$

$$\underbrace{(-1.1) \times \dots \times (-1.1)}_{9 \text{ times}} =$$

$$\underbrace{\left(\frac{2}{3}\right) \times \dots \times \left(\frac{2}{3}\right)}_{19 \text{ times}} =$$

$$\underbrace{\left(-\frac{11}{5}\right) \times \dots \times \left(-\frac{11}{5}\right)}_{\text{--- times}} = \left(-\frac{11}{5}\right)^x$$

$$\underbrace{(-12) \times \dots \times (-12)}_{\text{--- times}} = (-12)^{15}$$

$$\underbrace{a \times \dots \times a}_{m \text{ times}} =$$

2. Write an expression with  $(-1)$  as its base that will produce a positive product.
3. Write an expression with  $(-1)$  as its base that will produce a negative product.
4. Rewrite each number in exponential notation using 2 as the base.
 

8 =	16 =	32 =
64 =	128 =	256 =
5. Tim wrote 16 as  $(-2)^4$ . Is he correct?
6. Could  $-2$  be used as a base to rewrite 32? 64? Why or why not?

## Lesson 2: Multiplication of Numbers in Exponential Form

### Classwork

In general, if  $x$  is any number and  $m, n$  are positive integers, then

$$x^m \cdot x^n = x^{m+n}$$

because

$$x^m \times x^n = \underbrace{(x \cdots x)}_{m \text{ times}} \times \underbrace{(x \cdots x)}_{n \text{ times}} = \underbrace{(x \cdots x)}_{m+n \text{ times}} = x^{m+n}.$$

#### Exercise 1

$$14^{23} \times 14^8 =$$

#### Exercise 2

$$(-72)^{10} \times (-72)^{13} =$$

#### Exercise 3

$$5^{94} \times 5^{78} =$$

#### Exercise 4

$$(-3)^9 \times (-3)^5 =$$

#### Exercise 5

Let  $a$  be a number.

$$a^{23} \cdot a^8 =$$

#### Exercise 6

Let  $f$  be a number.

$$f^{10} \cdot f^{13} =$$

#### Exercise 7

Let  $b$  be a number.

$$b^{94} \cdot b^{78} =$$

#### Exercise 8

Let  $x$  be a positive integer. If  $(-3)^9 \times (-3)^x = (-3)^{14}$ , what is  $x$ ?

What would happen if there were more terms with the same base? Write an equivalent expression for each problem.

**Exercise 9**

$$9^4 \times 9^6 \times 9^{13} =$$

**Exercise 10**

$$2^3 \times 2^5 \times 2^7 \times 2^9 =$$

Can the following expressions be simplified? If so, write an equivalent expression. If not, explain why not.

**Exercise 11**

$$6^5 \times 4^9 \times 4^3 \times 6^{14} =$$

**Exercise 14**

$$2^4 \times 8^2 = 2^4 \times 2^6 =$$

**Exercise 12**

$$(-4)^2 \cdot 17^5 \cdot (-4)^3 \cdot 17^7 =$$

**Exercise 15**

$$3^7 \times 9 = 3^7 \times 3^2 =$$

**Exercise 13**

$$15^2 \cdot 7^2 \cdot 15 \cdot 7^4 =$$

**Exercise 16**

$$5^4 \times 2^{11} =$$

**Exercise 17**

Let  $x$  be a number. Simplify the expression of the following number:

$$(2x^3)(17x^7) =$$

**Exercise 18**

Let  $a$  and  $b$  be numbers. Use the distributive law to simplify the expression of the following number:

$$a(a + b) =$$

**Exercise 19**

Let  $a$  and  $b$  be numbers. Use the distributive law to simplify the expression of the following number:

$$b(a + b) =$$

**Exercise 20**

Let  $a$  and  $b$  be numbers. Use the distributive law to simplify the expression of the following number:

$$(a + b)(a + b) =$$

In general, if  $x$  is nonzero and  $m, n$  are positive integers, then

$$\frac{x^m}{x^n} = x^{m-n}, \text{ if } m > n.$$

**Exercise 21**

$$\frac{7^9}{7^6} =$$

**Exercise 23**

$$\frac{\left(\frac{8}{5}\right)^9}{\left(\frac{8}{5}\right)^2} =$$

**Exercise 22**

$$\frac{(-5)^{16}}{(-5)^7} =$$

**Exercise 24**

$$\frac{13^5}{13^4} =$$

**Exercise 25**

Let  $a, b$  be nonzero numbers. What is the following number?

$$\frac{\left(\frac{a}{b}\right)^9}{\left(\frac{a}{b}\right)^2} =$$

**Exercise 26**

Let  $x$  be a nonzero number. What is the following number?

$$\frac{x^5}{x^4} =$$

Can the following expressions be simplified? If yes, write an equivalent expression for each problem. If not, explain why not.

**Exercise 27**

$$\frac{2^7}{4^2} = \frac{2^7}{2^4} =$$

**Exercise 29**

$$\frac{3^5 \cdot 2^8}{3^2 \cdot 2^3} =$$

**Exercise 28**

$$\frac{3^{23}}{27} = \frac{3^{23}}{3^3} =$$

**Exercise 30**

$$\frac{(-2)^7 \cdot 95^5}{(-2)^5 \cdot 95^4} =$$

**Exercise 31**

Let  $x$  be a number. Simplify the expression of each of the following numbers:

a.  $\frac{5}{x^3}(3x^8) =$

b.  $\frac{5}{x^3}(-4x^6) =$

c.  $\frac{5}{x^3}(11x^4) =$

**Exercise 32**

Anne used an online calculator to multiply  $2,000,000,000 \times 2,000,000,000,000$ . The answer showed up on the calculator as  $4e + 21$ , as shown below. Is the answer on the calculator correct? How do you know?

2000000000 x 2000000000000 =

4e+21

Rad	⋮	x!	(	)	%	AC
Inv	sin	ln	7	8	9	÷
π	cos	log	4	5	6	×
e	tan	√	1	2	3	-
Ans	EXP	x <sup>y</sup>	0	.	=	+

**Problem Set**

1. A certain ball is dropped from a height of  $x$  feet. It always bounces up to  $\frac{2}{3}x$  feet. Suppose the ball is dropped from 10 feet and is caught exactly when it touches the ground after the 30<sup>th</sup> bounce. What is the total distance traveled by the ball? Express your answer in exponential notation.

Bounce	Computation of Distance Traveled in Previous Bounce	Total Distance Traveled (in feet)
1		
2		
3		
4		
30		
$n$		

2. If the same ball is dropped from 10 feet and is caught exactly at the highest point after the 25<sup>th</sup> bounce, what is the total distance traveled by the ball? Use what you learned from the last problem.
3. Let  $a$  and  $b$  be numbers and  $b \neq 0$ , and let  $m$  and  $n$  be positive integers. Simplify each of the following expressions as much as possible:

$(-19)^5 \cdot (-19)^{11} =$	$2.7^5 \times 2.7^3 =$
$\frac{7^{10}}{7^3} =$	$\left(\frac{1}{5}\right)^2 \cdot \left(\frac{1}{5}\right)^{15} =$
$\left(-\frac{9}{7}\right)^m \cdot \left(-\frac{9}{7}\right)^n =$	$\frac{ab^3}{b^2} =$

4. Let the dimensions of a rectangle be  $(4 \times (871209)^5 + 3 \times 49762105)$  ft. by  $(7 \times (871209)^3 - (49762105)^4)$  ft. Determine the area of the rectangle. No need to expand all the powers.
5. A rectangular area of land is being sold off in smaller pieces. The total area of the land is  $2^{15}$  square miles. The pieces being sold are  $8^3$  square miles in size. How many smaller pieces of land can be sold at the stated size? Compute the actual number of pieces.

## Lesson 3: Numbers in Exponential Form Raised to a Power

### Classwork

For any number  $x$  and any positive integers  $m$  and  $n$ ,

$$(x^m)^n = x^{mn}$$

because

$$\begin{aligned} (x^m)^n &= \underbrace{(x \cdot x \cdots x)}_{m \text{ times}}^n \\ &= \underbrace{(x \cdot x \cdots x)}_{m \text{ times}} \times \cdots \times \underbrace{(x \cdot x \cdots x)}_{m \text{ times}} \\ &\quad \underbrace{\hspace{10em}}_{n \text{ times}} \\ &= x^{mn}. \end{aligned}$$

#### Exercise 1

$$(15^3)^9 =$$

#### Exercise 3

$$(3.4^{17})^4 =$$

#### Exercise 2

$$((-2)^5)^8 =$$

#### Exercise 4

Let  $s$  be a number.

$$(s^{17})^4 =$$

#### Exercise 5

Sarah wrote  $(3^5)^7 = 3^{12}$ . Correct her mistake. Write an exponential expression using a base of 3 and exponents of 5, 7, and 12 that would make her answer correct.

#### Exercise 6

A number  $y$  satisfies  $y^{24} - 256 = 0$ . What equation does the number  $x = y^4$  satisfy?

For any numbers  $x$  and  $y$ , and positive integer  $n$ ,

$$(xy)^n = x^n y^n$$

because

$$\begin{aligned} (xy)^n &= \underbrace{(xy) \cdots (xy)}_{n \text{ times}} \\ &= \underbrace{(x \cdot x \cdots x)}_{n \text{ times}} \cdot \underbrace{(y \cdot y \cdots y)}_{n \text{ times}} \\ &= x^n y^n. \end{aligned}$$

**Exercise 7**

$$(11 \times 4)^9 =$$

**Exercise 8**

$$(3^2 \times 7^4)^5 =$$

**Exercise 9**

Let  $a$ ,  $b$ , and  $c$  be numbers.

$$(3^2 a^4)^5 =$$

**Exercise 10**

Let  $x$  be a number.

$$(5x)^7 =$$

**Exercise 11**

Let  $x$  and  $y$  be numbers.

$$(5xy^2)^7 =$$

**Exercise 12**

Let  $a$ ,  $b$ , and  $c$  be numbers.

$$(a^2 bc^3)^4 =$$

**Exercise 13**

Let  $x$  and  $y$  be numbers,  $y \neq 0$ , and let  $n$  be a positive integer. How is  $\left(\frac{x}{y}\right)^n$  related to  $x^n$  and  $y^n$ ?

**Problem Set**

1. Show (prove) in detail why  $(2 \cdot 3 \cdot 7)^4 = 2^4 3^4 7^4$ .
2. Show (prove) in detail why  $(xyz)^4 = x^4 y^4 z^4$  for any numbers  $x, y, z$ .
3. Show (prove) in detail why  $(xyz)^n = x^n y^n z^n$  for any numbers  $x, y$ , and  $z$  and for any positive integer  $n$ .

## Lesson 4: Numbers Raised to the Zeroth Power

### Classwork

For any numbers  $x$ ,  $y$ , and any positive integers  $m$ ,  $n$ , the following holds

$$x^m \cdot x^n = x^{m+n} \quad (1)$$

$$(x^m)^n = x^{mn} \quad (2)$$

$$(xy)^n = x^n y^n. \quad (3)$$

**Definition:** \_\_\_\_\_

### Exercise 1

List all possible cases of whole numbers  $m$  and  $n$  for identity (1). More precisely, when  $m > 0$  and  $n > 0$ , we already know that (1) is correct. What are the other possible cases of  $m$  and  $n$  for which (1) is yet to be verified?

### Exercise 2

Check that equation (1) is correct for each of the cases listed in Exercise 1.

**Exercise 3**

Do the same with equation (2) by checking it case-by-case.

**Exercise 4**

Do the same with equation (3) by checking it case-by-case.

**Exercise 5**

Write the expanded form of 8,374 using exponential notation.

**Exercise 6**

Write the expanded form of 6,985,062 using exponential notation.

## Problem Set

Let  $x, y$  be numbers ( $x, y \neq 0$ ). Simplify each of the following expressions of numbers.

1. $\frac{y^{12}}{y^{12}} =$	2. $9^{15} \cdot \frac{1}{9^{15}} =$
3. $(7(123456.789)^4)^0 =$	4. $2^2 \cdot \frac{1}{2^5} \cdot 2^5 \cdot \frac{1}{2^2} =$
5. $\frac{x^{41}}{y^{15}} \cdot \frac{y^{15}}{x^{41}} =$	

## Lesson 5: Negative Exponents and the Laws of Exponents

### Classwork

**Definition:** For any positive number  $x$  and for any positive integer  $n$ , we define  $x^{-n} = \frac{1}{x^n}$ .

Note that this definition of negative exponents says  $x^{-1}$  is just the reciprocal,  $\frac{1}{x}$ , of  $x$ .

As a consequence of the definition, for a positive  $x$  and all integers  $b$ , we get

$$x^{-b} = \frac{1}{x^b}.$$

### Exercise 1

Verify the general statement  $x^{-b} = \frac{1}{x^b}$  for  $x = 3$  and  $b = -5$ .

### Exercise 2

What is the value of  $(3 \times 10^{-2})$ ?

**Exercise 3**

What is the value of  $(3 \times 10^{-5})$ ?

**Exercise 4**

Write the complete expanded form of the decimal 4.728 in exponential notation.

For Exercises 5–10, write an equivalent expression, in exponential notation, to the one given and simplify as much as possible.

**Exercise 5**

$$5^{-3} =$$

**Exercise 6**

$$\frac{1}{8^9} =$$

**Exercise 7**

$$3 \cdot 2^{-4} =$$

**Exercise 8**

Let  $x$  be a nonzero number.

$$x^{-3} =$$

**Exercise 9**

Let  $x$  be a nonzero number.

$$\frac{1}{x^9} =$$

**Exercise 10**

Let  $x, y$  be two nonzero numbers.

$$xy^{-4} =$$

We accept that for positive numbers  $x$ ,  $y$  and all integers  $a$  and  $b$ ,

$$x^a \cdot x^b = x^{a+b}$$

$$(x^b)^a = x^{ab}$$

$$(xy)^a = x^a y^a.$$

We claim

$$\frac{x^a}{x^b} = x^{a-b} \quad \text{for all integers } a, b.$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} \quad \text{for any integer } a.$$

### Exercise 11

$$\frac{19^2}{19^5} =$$

### Exercise 12

$$\frac{17^{16}}{17^{-3}} =$$

### Exercise 13

If we let  $b = -1$  in (11),  $a$  be any integer, and  $y$  be any positive number, what do we get?

**Exercise 14**

Show directly that  $\left(\frac{7}{5}\right)^{-4} = \frac{7^{-4}}{5^{-4}}$ .

**Problem Set**

1. Compute:  $3^3 \times 3^2 \times 3^1 \times 3^0 \times 3^{-1} \times 3^{-2} =$

Compute:  $5^2 \times 5^{10} \times 5^8 \times 5^0 \times 5^{-10} \times 5^{-8} =$

Compute for a nonzero number,  $a$ :  $a^m \times a^n \times a^l \times a^{-n} \times a^{-m} \times a^{-l} \times a^0 =$

2. Without using (10), show directly that  $(17.6^{-1})^8 = 17.6^{-8}$ .

3. Without using (10), show (prove) that for any whole number  $n$  and any positive number  $y$ ,  $(y^{-1})^n = y^{-n}$ .

4. Show directly without using (13) that  $\frac{2.8^{-5}}{2.8^7} = 2.8^{-12}$ .

## Lesson 6: Proofs of Laws of Exponents

### Classwork

#### The Laws of Exponents

For  $x, y > 0$ , and all integers  $a, b$ , the following holds:

$$x^a \cdot x^b = x^{a+b}$$

$$(x^b)^a = x^{ab}$$

$$(xy)^a = x^a y^a$$

#### Facts we will use to prove (11):

(A) (11) is already known to be true when the integers  $a$  and  $b$  satisfy  $a \geq 0, b \geq 0$ .

(B)  $x^{-m} = \frac{1}{x^m}$  for any whole number  $m$ .

(C)  $\left(\frac{1}{x}\right)^m = \frac{1}{x^m}$  for any whole number  $m$ .

### Exercise 1

Show that (C) is implied by equation (5) of Lesson 4 when  $m > 0$ , and explain why (C) continues to hold even when  $m = 0$ .

**Exercise 2**

Show that **(B)** is in fact a special case of (11) by rewriting it as  $(x^m)^{-1} = x^{(-1)m}$  for any whole number  $m$ , so that if  $b = m$  (where  $m$  is a whole number) and  $a = -1$ , (11) becomes **(B)**.

**Exercise 3**

Show that **(C)** is a special case of (11) by rewriting **(C)** as  $(x^{-1})^m = x^{m(-1)}$  for any whole number  $m$ . Thus, **(C)** is the special case of (11) when  $b = -1$  and  $a = m$ , where  $m$  is a whole number.

**Exercise 4**

*Proof of Case (iii):* Show that when  $a < 0$  and  $b \geq 0$ ,  $(x^b)^a = x^{ab}$  is still valid. Let  $a = -c$  for some positive integer  $c$ . Show that the left side and right sides of  $(x^b)^a = x^{ab}$  are equal.

**Problem Set**

1. You sent a photo of you and your family on vacation to seven Facebook friends. If each of them sends it to five of their friends, and each of those friends sends it to five of their friends, and those friends send it to five more, how many people (not counting yourself) will see your photo? No friend received the photo twice. Express your answer in exponential notation.

# of New People to View Your Photo	Total # of People to View Your Photo

2. Show directly, without using (11), that  $(1.27^{-36})^{85} = 1.27^{-36 \cdot 85}$ .
3. Show directly that  $\left(\frac{2}{13}\right)^{-127} \cdot \left(\frac{2}{13}\right)^{-56} = \left(\frac{2}{13}\right)^{-183}$ .
4. Prove for any positive number  $x$ ,  $x^{-127} \cdot x^{-56} = x^{-183}$ .
5. Prove for any positive number  $x$ ,  $x^{-m} \cdot x^{-n} = x^{-m-n}$  for positive integers  $m$  and  $n$ .
6. Which of the preceding four problems did you find easiest to do? Explain.
7. Use the properties of exponents to write an equivalent expression that is a product of distinct primes, each raised to an integer power.

$$\frac{10^5 \cdot 9^2}{6^4} =$$

## Lesson 7: Magnitude

### Classwork

**Fact 1:** The number  $10^n$ , for arbitrarily large positive integers  $n$ , is a **big number** in the sense that given a number  $M$  (no matter how big it is) there is a power of 10 that exceeds  $M$ .

**Fact 2:** The number  $10^{-n}$ , for arbitrarily large positive integers  $n$ , is a **small number** in the sense that given a positive number  $S$  (no matter how small it is), there is a (negative) power of 10 that is smaller than  $S$ .

### Exercise 1

Let  $M = 993,456,789,098,765$ . Find the smallest power of 10 that will exceed  $M$ .

### Exercise 2

Let  $M = 78,491 \frac{899}{987}$ . Find the smallest power of 10 that will exceed  $M$ .

**Exercise 3**

Let  $M$  be a positive integer. Explain how to find the smallest power of 10 that exceeds it.

**Exercise 4**

The chance of you having the same DNA as another person (other than an identical twin) is approximately 1 in 10 trillion (one trillion is a 1 followed by 12 zeros). Given the fraction, express this very small number using a negative power of 10.

$$\frac{1}{10,000,000,000,000}$$

**Exercise 5**

The chance of winning a big lottery prize is about  $10^{-8}$ , and the chance of being struck by lightning in the U.S. in any given year is about 0.000001. Which do you have a greater chance of experiencing? Explain.

**Exercise 6**

There are about 100 million smartphones in the U.S. Your teacher has one smartphone. What share of U.S. smartphones does your teacher have? Express your answer using a negative power of 10.

**Problem Set**

1. What is the smallest power of 10 that would exceed 987,654,321,098,765,432?
2. What is the smallest power of 10 that would exceed 999,999,999,991?
3. Which number is equivalent to 0.0000001:  $10^7$  or  $10^{-7}$ ? How do you know?
4. Sarah said that 0.00001 is bigger than 0.001 because the first number has more digits to the right of the decimal point. Is Sarah correct? Explain your thinking using negative powers of 10 and the number line.
5. Place each of the following numbers on a number line in its approximate location:

$10^5$      $10^{-99}$      $10^{-17}$      $10^{14}$      $10^{-5}$      $10^{30}$

## Lesson 8: Estimating Quantities

### Classwork

#### Exercise 1

The Federal Reserve states that the average household in January of 2013 had \$7,122 in credit card debt. About how many times greater is the U.S. national debt, which is \$16,755,133,009,522? Rewrite each number to the nearest power of 10 that exceeds it, and then compare.

#### Exercise 2

There are about 3,000,000 students attending school, kindergarten through 12<sup>th</sup> grade, in New York. Express the number of students as a single-digit integer times a power of 10.

The average number of students attending a middle school in New York is  $8 \times 10^2$ . How many times greater is the overall number of K-12 students compared to the number of middle school students?

**Exercise 3**

A conservative estimate of the number of stars in the universe is  $6 \times 10^{22}$ . The average human can see about 3,000 stars at night with his naked eye. About how many times more stars are there in the universe, compared to the stars a human can actually see?

**Exercise 4**

The estimated world population in 2011 was  $7 \times 10^9$ . Of the total population, 682 million of those people were left-handed. Approximately what percentage of the world population is left-handed according to the 2011 estimation?

**Exercise 5**

The average person takes about 30,000 breaths per day. Express this number as a single-digit integer times a power of 10.

If the average American lives about 80 years (or about 30,000 days), how many total breaths will a person take in her lifetime?

**Problem Set**

1. The Atlantic Ocean region contains approximately  $2 \times 10^{16}$  gallons of water. Lake Ontario has approximately 8,000,000,000 gallons of water. How many Lake Ontarios would it take to fill the Atlantic Ocean region in terms of gallons of water?
2. U.S. national forests cover approximately 300,000 square miles. Conservationists want the total square footage of forests to be  $300,000^2$  square miles. When Ivanna used her phone to do the calculation, her screen showed the following:



- a. What does the answer on her screen mean? Explain how you know.
  - b. Given that the U.S. has approximately 4 million square miles of land, is this a reasonable goal for conservationists? Explain. .
3. The average American is responsible for about 20,000 kilograms of carbon emission pollution each year. Express this number as a single-digit integer times a power of 10.
  4. The United Kingdom is responsible for about  $1 \times 10^4$  kilograms. Which country is responsible for greater carbon emission pollution each year? By how much?

## Lesson 9: Scientific Notation

### Classwork

A positive, finite decimal  $s$  is said to be written in **scientific notation** if it is expressed as a product  $d \times 10^n$ , where  $d$  is a finite decimal so that  $1 \leq d < 10$ , and  $n$  is an integer.

The integer  $n$  is called the **order of magnitude** of the decimal  $d \times 10^n$ .

Are the following numbers written in scientific notation? If not, state why.

**Exercise 1**

$$1.908 \times 10^{17}$$

**Exercise 4**

$$4.0701 + 10^7$$

**Exercise 2**

$$0.325 \times 10^{-2}$$

**Exercise 5**

$$18.432 \times 5^8$$

**Exercise 3**

$$7.99 \times 10^{32}$$

**Exercise 6**

$$8 \times 10^{-11}$$

Use the table below to complete Exercises 7 and 8.

The table below shows the debt of the three most populous states and the three least populous states.

State	Debt (in dollars)	Population (2012)
California	407,000,000,000	38,000,000
New York	337,000,000,000	19,000,000
Texas	276,000,000,000	26,000,000
North Dakota	4,000,000,000	69,000
Vermont	4,000,000,000	62,600
Wyoming	2,000,000,000	57,600

### Exercise 7

- a. What is the sum of the debts for the three most populous states? Express your answer in scientific notation.

- b. What is the sum of the debt for the three least populous states? Express your answer in scientific notation.

- c. How much larger is the combined debt of the three most populous states than that of the three least populous states? Express your answer in scientific notation.

### Exercise 8

- a. What is the sum of the population of the three most populous states? Express your answer in scientific notation.
- b. What is the sum of the population of the three least populous states? Express your answer in scientific notation.
- c. Approximately how many times greater is the total population of California, New York, and Texas compared to the total population of North Dakota, Vermont, and Wyoming?

**Exercise 9**

All planets revolve around the sun in elliptical orbits. Uranus's furthest distance from the sun is approximately  $3.004 \times 10^9$  km, and its closest distance is approximately  $2.749 \times 10^9$  km. Using this information, what is the average distance of Uranus from the sun?

**Problem Set**

1. Write the number 68,127,000,000,000 in scientific notation. Which of the two representations of this number do you prefer? Explain.
2. Here are the masses of the so-called inner planets of the Solar System.  
Mercury:  $3.3022 \times 10^{23}$  kg                      Earth:  $5.9722 \times 10^{24}$  kg  
Venus:  $4.8685 \times 10^{24}$  kg                      Mars:  $6.4185 \times 10^{23}$  kg

What is the average mass of all four inner planets? Write your answer in scientific notation.

## Lesson 10: Operations with Numbers in Scientific Notation

### Classwork

#### Exercise 1

The speed of light is 300,000,000 meters per second. The sun is approximately  $1.5 \times 10^{11}$  meters from Earth. How many seconds does it take for sunlight to reach Earth?

#### Exercise 2

The mass of the moon is about  $7.3 \times 10^{22}$  kg. It would take approximately 26,000,000 moons to equal the mass of the sun. Determine the mass of the sun.

**Exercise 3**

The mass of Earth is  $5.9 \times 10^{24}$  kg. The mass of Pluto is 13,000,000,000,000,000,000 kg. Compared to Pluto, how much greater is Earth's mass than Pluto's mass?

**Exercise 4**

Using the information in Exercises 2 and 3, find the combined mass of the moon, Earth, and Pluto.

**Exercise 5**

How many combined moon, Earth, and Pluto masses (i.e., the answer to Exercise 4) are needed to equal the mass of the sun (i.e., the answer to Exercise 2)?

**Problem Set**

1. The sun produces  $3.8 \times 10^{27}$  joules of energy per second. How much energy is produced in a year? (Note: a year is approximately 31,000,000 seconds).
2. On average, Mercury is about 57,000,000 km from the sun, whereas Neptune is about  $4.5 \times 10^9$  km from the sun. What is the difference between Mercury's and Neptune's distances from the sun?
3. The mass of Earth is approximately  $5.9 \times 10^{24}$  kg, and the mass of Venus is approximately  $4.9 \times 10^{24}$  kg.
  - a. Find their combined mass.
  - b. Given that the mass of the sun is approximately  $1.9 \times 10^{30}$  kg, how many Venuses and Earths would it take to equal the mass of the sun?



**Exercise 4**

Compute how many times heavier a proton is than an electron (that is, find the value of the ratio). Round your final answer to the nearest one.

**Example 2**

The U.S. national debt as of March 23, 2013, rounded to the nearest dollar, is \$16,755,133,009,522. According to the 2012 U.S. census, there are about 313,914,040 U.S. citizens. What is each citizen's approximate share of the debt?

$$\begin{aligned}\frac{1.6755 \times 10^{13}}{3.14 \times 10^8} &= \frac{1.6755}{3.14} \times \frac{10^{13}}{10^8} \\ &= \frac{1.6755}{3.14} \times 10^5 \\ &= 0.533598... \times 10^5 \\ &\approx 0.5336 \times 10^5 \\ &= 53360\end{aligned}$$

Each U.S. citizen's share of the national debt is about \$53,360.

**Exercise 5**

The geographic area of California is 163,696 sq. mi., and the geographic area of the U.S. is 3,794,101 sq. mi. Let's round off these figures to  $1.637 \times 10^5$  and  $3.794 \times 10^6$ . In terms of area, roughly estimate how many Californias would make up one U.S. Then compute the answer to the nearest ones.

**Exercise 6**

The average distance from Earth to the moon is about  $3.84 \times 10^5$  km, and the distance from Earth to Mars is approximately  $9.24 \times 10^7$  km in year 2014. On this simplistic level, how much further is traveling from Earth to Mars than from Earth to the moon?

**Problem Set**

1. There are approximately  $7.5 \times 10^{18}$  grains of sand on Earth. There are approximately  $7 \times 10^{27}$  atoms in an average human body. Are there more grains of sand on Earth or atoms in an average human body? How do you know?
2. About how many times more atoms are in a human body compared to grains of sand on Earth?
3. Suppose the geographic areas of California and the US are  $1.637 \times 10^5$  and  $3.794 \times 10^6$  sq. mi., respectively. California's population (as of 2012) is approximately  $3.804 \times 10^7$  people. If population were proportional to area, what would be the U.S. population?
4. The actual population of the U.S. (as of 2012) is approximately  $3.14 \times 10^8$ . How does the population density of California (i.e., the number of people per sq. mi.) compare with the population density of the U.S.?

## Lesson 12: Choice of Unit

### Classwork

#### Exercise 1

A certain brand of MP3 player will display how long it will take to play through its entire music library. If the maximum number of songs the MP3 player can hold is 1,000 (and the average song length is 4 minutes), would you want the time displayed in terms of seconds-, days-, or years-worth of music? Explain.

#### Exercise 2

You have been asked to make frosted cupcakes to sell at a school fundraiser. Each frosted cupcake contains about 20 grams of sugar. Bake sale coordinators expect 500 people will attend the event. Assume everyone who attends will buy a cupcake; does it make sense to buy sugar in grams, pounds, or tons? Explain.

#### Exercise 3

The seafloor spreads at a rate of approximately 10 cm per year. If you were to collect data on the spread of the seafloor each week, which unit should you use to record your data? Explain.

Gigaelectronvolt,  $\frac{\text{GeV}}{c^2}$ , is what particle physicists use as the unit of mass.

$$1 \text{ Gigaelectronvolt} = 1.783 \times 10^{-27} \text{ kg}$$

$$\text{Mass of 1 proton} = 1.672\,622 \times 10^{-27} \text{ kg}$$

#### Exercise 4

Show that the mass of a proton is  $0.938 \frac{\text{GeV}}{c^2}$ .

In popular science writing, a commonly used unit is the light-year, or the distance light travels in one year (note: one year is defined as 365.25 days).

$$1 \text{ light – year} = 9,460,730,472,580.800 \text{ km} \approx 9.46073 \times 10^{12} \text{ km}$$

#### Exercise 5

The distance of the nearest star (*Proxima Centauri*) to the sun is approximately  $4.013\,336\,473 \times 10^{13}$  km. Show that *Proxima Centauri* is 4.2421 light-years from the sun.

**Exploratory Challenge 2**

Suppose you are researching atomic diameters and find that credible sources provided the diameters of five different atoms as shown in the table below. All measurements are in cm.

$1 \times 10^{-8}$	$1 \times 10^{-12}$	$5 \times 10^{-8}$	$5 \times 10^{-10}$	$5.29 \times 10^{-11}$
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**Exercise 6**

What new unit might you introduce in order to discuss the differences in diameter measurements?

**Exercise 7**

Name your unit and explain why you chose it.

**Exercise 8**

Using the unit you have defined, rewrite the five diameter measurements.

### Problem Set

- Verify the claim that, in terms of gigaelectronvolts, the mass of an electron is 0.000511.
- The maximum distance between Earth and the sun is  $1.52098232 \times 10^8$  km, and the minimum distance is  $1.47098290 \times 10^8$  km.<sup>1</sup> What is the average distance between Earth, and the sun in scientific notation?
- Suppose you measure the following masses in terms of kilograms:

$2.6 \times 10^{21}$	$9.04 \times 10^{23}$
$8.82 \times 10^{23}$	$2.3 \times 10^{18}$
$1.8 \times 10^{12}$	$2.103 \times 10^{22}$
$8.1 \times 10^{20}$	$6.23 \times 10^{18}$
$6.723 \times 10^{19}$	$1.15 \times 10^{20}$
$7.07 \times 10^{21}$	$7.210 \times 10^{29}$
$5.11 \times 10^{25}$	$7.35 \times 10^{24}$
$7.8 \times 10^{19}$	$5.82 \times 10^{26}$

What new unit might you introduce in order to aid discussion of the masses in this problem? Name your unit and express it using some power of 10. Rewrite each number using your newly defined unit.

<sup>1</sup> Note: Earth's orbit is elliptical, not circular.

## Lesson 13: Comparison of Numbers Written in Scientific Notation and Interpreting Scientific Notation Using Technology

### Classwork

There is a general principle that underlies the comparison of two numbers in scientific notation: *Reduce everything to whole numbers if possible.* To this end, we recall two basic facts.

1. **Inequality (A)** in Lesson 7: Let  $x$  and  $y$  be numbers and let  $z > 0$ . Then  $x < y$  if and only if  $xz < yz$ .
2. Comparison of whole numbers:
  - a. If two whole numbers have different numbers of digits, then the one with more digits is greater.
  - b. Suppose two whole numbers  $p$  and  $q$  have the same number of digits and, moreover, they agree digit-by-digit (starting from the left) until the  $n^{\text{th}}$  place. If the digit of  $p$  in the  $(n + 1)^{\text{th}}$  place is greater than the corresponding digit in  $q$ , then  $p > q$ .

### Exercise 1

The Fornax Dwarf galaxy is  $4.6 \times 10^5$  light-years away from Earth, while Andromeda I is  $2.430 \times 10^6$  light-years away from Earth. Which is closer to Earth?

### Exercise 2

The average lifetime of the tau lepton is  $2.906 \times 10^{-13}$  seconds, and the average lifetime of the neutral pion is  $8.4 \times 10^{-17}$  seconds. Explain which subatomic particle has a longer average lifetime.

**Exploratory Challenge 1/Exercise 3**

**Theorem:** Given two numbers in scientific notation,  $a \times 10^m$  and  $b \times 10^n$ , if  $m < n$ , then  $a \times 10^m < b \times 10^n$ .

Prove the theorem.

**Exercise 4**

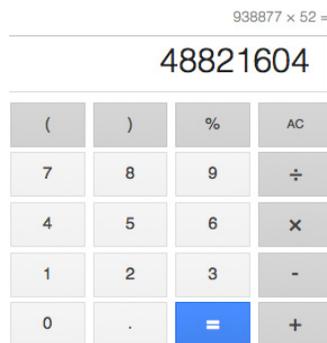
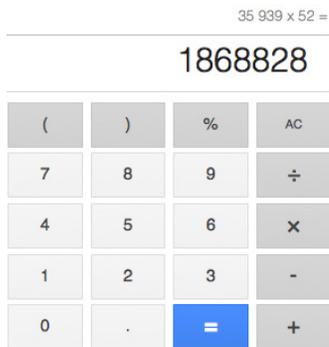
Compare  $9.3 \times 10^{28}$  and  $9.2879 \times 10^{28}$ .

**Exercise 5**

Chris said that  $5.3 \times 10^{41} < 5.301 \times 10^{41}$  because 5.3 has fewer digits than 5.301. Show that even though his answer is correct, his reasoning is flawed. Show him an example to illustrate that his reasoning would result in an incorrect answer. Explain.

**Exploratory Challenge 2/Exercise 6**

You have been asked to determine the exact number of Google searches that are made each year. The only information you are provided is that there are 35,939,938,877 searches performed each week. Assuming the exact same number of searches is performed each week for the 52 weeks in a year, how many total searches will have been performed in one year? Your calculator does not display enough digits to get the exact answer. Therefore, you must break down the problem into smaller parts. Remember, you cannot approximate an answer because you need to find an exact answer. Use the screen shots below to help you reach your answer.



Yahoo is another popular search engine. Yahoo receives requests for 1,792,671,355 searches each month. Assuming the same number of searches is performed each month, how many searches are performed on Yahoo each year? Use the screen shots below to help determine the answer.

1792 x 12 =

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8056020			
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7	8	9	÷
4	5	6	×
1	2	3	-
0	.	=	+

**Problem Set**

- Write out a detailed proof of the fact that, given two numbers in scientific notation,  $a \times 10^n$  and  $b \times 10^n$ ,  $a < b$ , if and only if  $a \times 10^n < b \times 10^n$ .
  - Let  $A$  and  $B$  be two positive numbers, with no restrictions on their size. Is it true that  $A \times 10^{-5} < B \times 10^5$ ?
  - Now, if  $A \times 10^{-5}$  and  $B \times 10^5$  are written in scientific notation, is it true that  $A \times 10^{-5} < B \times 10^5$ ? Explain.
- The mass of a neutron is approximately  $1.674927 \times 10^{-27}$  kg. Recall that the mass of a proton is  $1.672622 \times 10^{-27}$  kg. Explain which is heavier.
- The average lifetime of the Z boson is approximately  $3 \times 10^{-25}$  seconds, and the average lifetime of a neutral rho meson is approximately  $4.5 \times 10^{-24}$  seconds.
  - Without using the theorem from today's lesson, explain why the neutral rho meson has a longer average lifetime.
  - Approximately how much longer is the lifetime of a neutral rho meson than a Z boson?