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¹ Each lesson is ONE day, and ONE day is considered a 45-minute period.

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Geometry • Module 5

Circles With and Without Coordinates

OVERVIEW

With geometric intuition well established through Modules 1, 2, 3, and 4, students are now ready to explore the rich geometry of circles. This module brings together the ideas of similarity and congruence studied in Modules 1 and 2, the properties of length and area studied in Modules 3 and 4, and the work of geometric construction studied throughout the entire year. It also includes the specific properties of triangles, special quadrilaterals, parallel lines and transversals, and rigid motions established and built upon throughout this mathematical story.

This module's focus is on the possible geometric relationships between a pair of intersecting lines and a circle drawn on the page. If the lines are perpendicular and one passes through the center of the circle, then the relationship encompasses the perpendicular bisectors of chords in a circle and the association between a tangent line and a radius drawn to the point of contact. If the lines meet at a point on the circle, then the relationship involves inscribed angles. If the lines meet at the center of the circle, then the relationship involves central angles. If the lines meet at a different point inside the circle or at a point outside the circle, then the relationship includes the secant angle theorems and tangent angle theorems.

Topic A, through a hands-on activity, leads students first to Thales' theorem (an angle drawn from a diameter of a circle to a point on the circle is sure to be a right angle), then to possible converses of Thales' theorem, and finally to the general inscribed-central angle theorem. Students use this result to solve unknown angle problems. Through this work, students construct triangles and rectangles inscribed in circles and study their properties (**G-C.A.2, G-C.A.3**).

Topic B defines the measure of an arc and establishes results relating chord lengths and the measures of the arcs they subtend. Students build on their knowledge of circles from Module 2 and prove that all circles are similar. Students develop a formula for arc length in addition to a formula for the area of a sector and practice their skills solving unknown area problems (**G-C.A.1, G-C.A.2, G-C.B.5**).

In Topic C, students explore geometric relations in diagrams of two secant lines, or a secant and tangent line (possibly even two tangent lines), meeting a point inside or outside of a circle. They establish the secant angle theorems and tangent-secant angle theorems. By drawing auxiliary lines, students also notice similar triangles and thereby discover relationships between lengths of line segments appearing in these diagrams (**G-C.A.2, G-C.A.3, G-C.A.4**).

Topic D brings in coordinate geometry to establish the equation of a circle. Students solve problems to find the equations of specific tangent lines or the coordinates of specific points of contact. They also express circles via analytic equations (**G-GPE.A.1, G-GPE.B.4**).

The module concludes with Topic E focusing on the properties of quadrilaterals inscribed in circles and establishing Ptolemy's theorem. This result codifies the Pythagorean theorem, curious facts about triangles, properties of the regular pentagon, and trigonometric relationships. It serves as a final unifying flourish for students' year-long study of geometry (**G-C.A.3**).

Focus Standards

Understand and apply theorems about circles.

- G-C.A.1** Prove² that all circles are similar.
- G-C.A.2** Identify and describe relationships among inscribed angles, radii, and chords. *Include³ the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*
- G-C.A.3** Construct the inscribed and circumscribed circles of a triangle, and prove² properties of angles for a quadrilateral inscribed in a circle.

Find arc lengths and areas of sectors of circles.

- G-C.B.5** Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Translate between the geometric description and the equation for a conic section.

- G-GPE.A.1** Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Use coordinates to prove simple geometric theorems algebraically.

- G-GPE.B.4** Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point $(1, \sqrt{3})$ lies on the circle centered at the origin and containing the point $(0, 2)$.*

Extension Standards

Apply trigonometry to general triangles.

- G-SRT.D.9** (+) Derive the formula $A = 1/2 ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

Understand and apply theorems about circles.

- G-C.A.4** (+) Construct a tangent line from a point outside a given circle to the circle.

² Prove *and apply* (in preparation for Regents Exams).

³ Include angles formed by secants (in preparation for Regents Exams).

Foundational Standards

Understand and apply the Pythagorean Theorem.

- 8.G.B.7** Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
- 8.G.B.8** Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

Experiment with transformations in the plane.

- G-CO.A.3** Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
- G-CO.A.5** Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Prove geometric theorems.

- G-CO.C.9** Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*
- G-CO.C.10** Prove² theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180° ; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*
- G-CO.C.11** Prove² theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

Make geometric constructions.

- G-CO.D.12** Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*

Prove theorems involving similarity.

- G-SRT.B.5** Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Focus Standards for Mathematical Practice

- MP.1** **Make sense of problems and persevere in solving them.** Students solve a number of complex unknown angles and unknown area geometry problems, work to devise the geometric construction of given objects, and adapt established geometric results to new contexts and to new conclusions.
- MP.3** **Construct viable arguments and critique the reasoning of others.** Students must provide justification for the steps in geometric constructions and the reasoning in geometric proofs, as well as create their own proofs of results and their extensions.
- MP.7** **Look for and make use of structure.** Students must identify features within complex diagrams (e.g., similar triangles, parallel chords, and cyclic quadrilaterals) which provide insight as to how to move forward with their thinking.

Terminology

New or Recently Introduced Terms

- **Arc Length** (The *length of an arc* is the circular distance around the arc.)
- **Central Angle** (A *central angle* of a circle is an angle whose vertex is the center of a circle.)
- **Chord** (Given a circle C , let P and Q be points on C . The \overline{PQ} is called a *chord* of C .)
- **Cyclic Quadrilateral** (A quadrilateral inscribed in a circle is called a *cyclic quadrilateral*.)
- **Inscribed Angle** (An *inscribed angle* is an angle whose vertex is on a circle, and each side of the angle intersects the circle in another point.)
- **Inscribed Polygon** (A polygon is *inscribed* in a circle if all vertices of the polygon lie on the circle.)
- **Secant Line** (A *secant line* to a circle is a line that intersects a circle in exactly two points.)
- **Sector** (Let AB be an arc of a circle. The *sector of a circle with arc AB* is the union of all radii of the circle that have an endpoint in arc AB . The arc AB is called the *arc of the sector*, and the length of any radius of the circle is called the *radius of the sector*.)
- **Tangent Line** (A *tangent line to a circle* is a line in the same plane that intersects the circle in one and only one point. This point is called the *point of tangency*.)

Familiar Terms and Symbols⁴

- Circle
- Diameter
- Radius

⁴ These are terms and symbols students have seen previously.

Suggested Tools and Representations

- Compass and straightedge
- Geometer's Sketchpad or Geogebra Software
- White and colored paper, markers

Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic B	Constructed response with rubric	G-C.A.1, G-C.A.2, G-C.A.3, G-C.B.5
End-of-Module Assessment Task	After Topic D	Constructed response with rubric	G-C.A.1, G-C.A.2, G-C.A.3, G-GPE.A.1, G-GPE.B.4